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B.A./B.Sc. SECOND SEMESTER EXAMINATION, MAY 2016
FIRST YEAR [BATCH 2015-18]
MATHEMATICS (Honours)
Paper : II

Date : 18/05/2016
Time : 11 am – 3 pm
Full Marks : 100

[Use a separate Answer Book for each group]

Group – A

Answer any five questions from question no. 1 to 8 : [5×5]

1. If a, b c are positive rational numbers, prove that
$$a^a b^b c^c \ge \left(\frac{a+b}{2}\right)^{\frac{a+b}{2}} \left(\frac{b+c}{2}\right)^{\frac{b+c}{2}} \left(\frac{c+a}{2}\right)^{\frac{c+a}{2}} \ge \left(\frac{a+b+c}{3}\right)^{a+b+c}.$$ [5]

2. Prove that the minimum value of $x^2+y^2+z^2$ is $\left(\frac{c}{7}\right)^2$, where x, y, z are positive real numbers subject to the condition $2x+3y+6z=c$, c being a constant. Find the values of x, y, z for which the minimum value is attained. [5]

3. Show that $(\sqrt{2})^{\sqrt{3}}$ has infinitely many values. Show also that the points representing the general values lie on a circle in the complex plane. [2+3]

4. Prove that the equation $(x+1)^4 = a(x^4+1)$ is a reciprocal equation if $a \ne 1$ and solve it when $a=-2$. [1+4]

5. a) Find the nature of the roots of the equation $x^7+x^5-x^3=0$. [2]
   b) α, β, γ are the roots of the equation $x^3+px^2+qx+r=0$ $(r \ne 0)$. Find the equation whose roots are $\alpha-\frac{\beta\gamma}{\alpha}$, $\beta-\frac{\gamma\alpha}{\beta}$, $\gamma-\frac{\alpha\beta}{\gamma}$. [3]

6. Find the special roots of the equation $x^{15}-1=0$. Deduce that $2\cos\frac{2\pi}{15}$, $2\cos\frac{4\pi}{15}$, $2\cos\frac{8\pi}{15}$, $2\cos\frac{16\pi}{15}$ are the roots of the equation $x^4-x^3-4x^2+4x+1=0$. [5]

7. Solve the equation $x^3-15x^2-33x+847=0$ by Cardon's method. [5]

8. Solve the equation $x^4+4x^3-6x^2+20x+8=0$ by Ferrari's method. [5]

Answer any five questions from question no. 9 to 16 : [5×5]

9. Show that the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ is convergent. Find a rearrangement of the above series that is divergent. [1+4]

10. a) A function $f:\mathbb{R}\to\mathbb{R}$ is continuous on $\mathbb{R}$ and f(x) = 0 for all $x \in \mathbb{Q}$. Prove that f(x) = 0 for all $x \in \mathbb{R}$. [2]
    b) Prove that the series $\left(\frac{1}{2}\right)^p + \left(\frac{1.3}{2.4}\right)^p + \left(\frac{1.3.5}{2.4.6}\right)^p + \dots$ is convergent for p > 2 and divergent for $p \le 2$. [3]

11. a) If K_1 and K_2 are disjoint non-empty compact sets, show that there exists $\lambda_i \in K_i$ such that $0 < |\lambda_1 - \lambda_2| = \inf\{|x_1 - x_2| : x_i \in K_i\}$, i = 1, 2. [3]

b) Let $f:[-1,1] \to \mathbb{R}$ be defined by $f(x) = x\sin\frac{1}{x}, x \neq 0$

$$= \quad 0 \quad , x = 0$$

Prove that f is uniformly continuous on [–1, 1]. [2]

12. a) Let $f:[0,1] \to [0,1]$ be continuous on [0,1]. Prove that there exists a point α in [0, 1] such that $f(\alpha) = \alpha$. [2]

b) A function $f:[a,b] \to \mathbb{R}$ is continuous on [a, b] and $x_1, x_2, ..., x_n \in [a,b]$. Prove that there is a point $c \in [a,b]$ such that $f(c) = \frac{f(x_1) + f(x_2) + ... + f(x_n)}{n}$. [3]

13. a) A function $f:\mathbb{R} \to \mathbb{R}$ satisfies the condition $|f(x) - f(y)| \leq (x-y)^2$ for all $x, y \in \mathbb{R}$. Prove that f is a constant function. [2]

b) If $f''(x) \geq 0$ on [a,b], prove that $f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{1}{2}[f(x_1) + f(x_2)]$ for any two points x_1, x_2 in [a, b]. [3]

14. Let $f:[a,b] \to \mathbb{R}$ be differentiable on [a,b], $a, b \in \mathbb{R}$ and a < b. Let $f'(a) < f'(b)$ and k be any real number between $f'(a)$ and $f'(b)$. Prove that there exists $c \in (a,b)$ such that $f'(c) = k$. [5]

15. a) If f(x) = sinx, prove that $\lim_{h \to 0} \theta = \frac{1}{\sqrt{3}}$, where θ is given by $f(h) = f(0) + hf'(\theta h)$, $0 < \theta < 1$. [3]

b) Test the convergence of the series $1 + \frac{x}{1!} + \frac{2^2x^2}{2!} + \frac{3^3x^3}{3!} + ...$, x > 0. [2]

16. a) Evaluate : $\lim_{n \to \infty}\left(1 - \frac{1}{2n}\right)^{n+1}$. [3]

b) Let $f(x) = |x-1| + |x-2|$, $x \in [0,3]$. Show that f has a local minimum value at x = 1. [2]

Group – B

Answer any two questions from question no. 17 to 19 : [2×10]

17. a) Prove that $\begin{vmatrix} 1+a_1 & 1 & 1 & 1 \\ 1 & 1+a_2 & 1 & 1 \\ 1 & 1 & 1+a_3 & 1 \\ 1 & 1 & 1 & 1+a_4 \end{vmatrix} = a_1a_2a_3a_4\left(1 + \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4}\right)$. [3]

b) Prove that there exists orthogonal matrix P such that A = BP, where A and B are non-singular matrices such that $AA^t = BB^t$. [2]

c) Find the row reduced echelon matrix of $A = \begin{pmatrix} 2 & 0 & 4 & 2 \\ 3 & 2 & 6 & 5 \\ 5 & 2 & 10 & 7 \\ 0 & 3 & 2 & 5 \end{pmatrix}$. [5]

18. a) Let $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ be a basis of a vector space V of a field F and a non-zero vector β of V is expressed as $\beta = c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + c_4\alpha_4 + c_5\alpha_5$; $c_i \in F$. If $c_3 \neq 0$ then prove that $\{\alpha_1, \alpha_2, \beta, \alpha_4, \alpha_5\}$ is a new basis of V. [5]

b) Find the dimension of $\mathbb{C}$ over $\mathbb{C}$. [2]

c) Find a basis for $\mathbb{R}^3$ containing (1,3,1) and (1,5,4). [3]

19. a) Let W_1 and W_2 be subspaces of a vector space V such that $W_1 + W_2 = V$ and $W_1 \cap W_2 = \{\theta\}$. Prove that for each $v \in V$, there are unique vectors $v_1 \in W_1$ and $v_2 \in W_2$ such that $v = v_1 + v_2$. [2]

b) Let $\mathbb{R}_{2\times 2}$ be the real vector space of all 2×2 matrices over $\mathbb{R}$. Let $S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}_{2\times 2} \ : a+b=0 \right\}$. Prove that S is a subspace of $\mathbb{R}_{2\times 2}$. Find a basis and the dimension of S. [2+2+1]

c) Let $S = \{(x,y,z,w) \in \mathbb{R}^4 : x+y+z+w=0\}$ and $T = \{(x,y,z,w) \in \mathbb{R}^4 : x+3y-5z+6w=0\}$. Are S and T subspaces of $\mathbb{R}^4$? If so, find their basis. [3]

Answer <u>any three</u> questions from question no. <u>20 to 24</u> : [3×10]

20. a) Prove that if for a basic feasible solution x_B of a linear programming problem

Maximize $z = cx$

subject to $Ax = b$, $x \geq 0$

we have $z_j - c_j \geq 0$ for every column a_j of A, then x_B is an optimal solution. [6]

b) Find all the basic solutions of the following equations identifying in each case the basic vectors and the basic variables :

$x_1 + x_2 + x_3 = 4$

$2x_1 + 5x_2 - 2x_3 = 3$. [4]

21. a) Use Big-M method to maximize $z = 6x_1 + 4x_2$ subject to the constraints

$$2x_1 + 3x_2 \leq 30$$
$$3x_1 + 2x_2 \leq 24$$
$$x_1 + x_2 \geq 3$$
$$x_1, x_2 \geq 0$$

show that the solution is not unique. Find two solutions. [7]

b) Prove that the set of all feasible solutions of a linear programming problem is a convex set. [3]

22. a) Write down the dual of the following problem and solving the dual problem by simplex method, find the optimal solutions and values of the primal and dual as well :

Maximize $z = 3x_1 + 4x_2$

subject to
$$x_1 + x_2 \leq 10$$
$$2x_1 + 3x_2 \leq 18$$
$$x_1 \leq 8$$
$$x_2 \leq 6$$
$$x_1, x_2 \geq 0$$
[2+2+2+1]

b) Formulate Mathematically a Transportation problem (balanced) as an L.P.P having m origins and n destinations $(m, n \geq 2)$. [2]

c) What is the number of independent constraints in a balanced Transportation problem? [1]

23. a) In a Transportation problem the cost matrix is

	D_1	D_2	D_3	D_4	a_i
O_1	2	5	0	–3	12
O_2	4	6	8	1	15
O_3	4	0	4	5	14
O_4	2	6	1	4	9
b_j	10	8	12	20	50 \ 50

A initial Basic feasible solution is given by : $x_{14} = 12$, $x_{21} = 7$, $x_{24} = 8$, $x_{31} = 3$, $x_{32} = 8$, $x_{33} = 3$, $x_{43} = 9$.

Is this solution optimal and unique? Find the minimum cost of the problem. [2+2+1]

b) A machine operator processes five types of items on his machine each week and must choose a sequence for them. The set-up cost per change depends on the item presently on the machine and the set-up to be made according to the following table :

To Item

From Item	A	B	C	D	E
A	–	4	7	3	4
B	4	–	6	3	4
C	7	6	–	7	5
D	3	3	7	–	7
E	4	4	5	7	–

If he processes each type of item once and only once each week, how should he sequence the items on his machine in order to minimize the total set-up cost. [5]

24. a) $x_1 = 1$, $x_2 = 3$, $x_3 = 2$ is a feasible solution of the equations

$$2x_1 + 4x_2 - 2x_3 = 10$$

$$10x_1 + 3x_2 + 7x_3 = 33$$

Reduce the above F.S to B.F.S. [4]

b) Prove that if x be any feasible solution to the primal problem maximize z = cx subject to $Ax \le b$, $x \ge 0$ and v be any feasible solution to the corresponding dual, then $cx \le b'v$. [3]

c) Find the minimum cost solution for the following assignment problem : [3]

	I	II	III	IV
1	4	5	3	2
2	1	4	–2	3
3	4	2	1	–5

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